Examples of PDEs all whose points are characteristic

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$\tilde{X}$ natural lift of a vector field $X \in \mathfrak{X}(M)$ to FM,
$\tilde{X}^{(1)}$ natural lift of $\tilde{X}$ to $J^1(FM)$ by infinitesimal contact transformations.

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We only consider $\mathcal{L}^1_{12}$,$\mathcal{L}^1_{23}$, $\dim M=m\geq 3$.

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For the Lagrangian $\mathcal{L}^1_{12}$ we obtain a similar result with $3(m-1)$ equations.

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Euler-Lagrange equations of $\mathcal{L}_{23}^1$:

$$\left(f_a^k \frac{\partial f_i^h}{\partial x^k} - f_i^k \frac{\partial f_a^h}{\partial x^k} \right) f_h{}^1 = 0, \qquad \begin{array}{l} 4 \leq i \leq m, \\ a = 2, 3, \end{array}$$

$$2\left(f_1^k \frac{\partial f_a^h}{\partial x^k} - f_a^k \frac{\partial f_1^h}{\partial x^k} \right) f_h{}^1 + \sum_{b=4}^{m} \left(f_b^k \frac{\partial f_a^h}{\partial x^k} - f_a^k \frac{\partial f_b^h}{\partial x^k} \right) f_h{}^b = 0, \qquad a = 2, 3,$$

$$\left(f_2^k \frac{\partial f_3^h}{\partial x^k} - f_3^k \frac{\partial f_2^h}{\partial x^k} \right) f_h^j = 0, \qquad j \neq 2, 3.$$

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\varphi^{i-3}(j^1_x s) &= \omega^1([X_3, X_i]_x), \\
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3. φ is a submersion. Hence, R^1 is a fibred submanifold of $J^1(FM)$.

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where $4 \le i \le m$, $w \in T^*_x$ and $v \in \oplus^m T_x$.

Cauchy-Kowaleska PDEs system

First-order PDEs system of the Cauchy-Kowaleska type:

$$\frac{\partial u^\beta}{\partial x^m} = \Phi^\beta\left(x^i, u^\alpha, \frac{\partial u^\alpha}{\partial x^h}\right), \qquad \begin{array}{l} 1 \le i \le m, \\ 1 \le h \le m-1, \\ \alpha, \beta = 1 \dots n. \end{array}$$

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Otherwise, x_0 is characteristic.

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- We show that every hypersurface is characteristic for the PDEs of the extremals of $\mathcal{L}^1_{23}$.

GOAL

- We extend the notion of a characteristic submanifold to the case of underdetermined PDEs systems, that is, $R^1 \subset J^1 P$ being a PDEs system of codimension $\nu \leq n$.
- We show that every hypersurface is characteristic for the PDEs of the extremals of $\mathcal{L}^1_{23}$.
- This system is proved to be formally integrable.

Characteristic submanifold for an arbitrary PDEs system

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Recall

$$0 \to V_{j^1_{x_0} s}(p^1_0) \xrightarrow{\varepsilon} T_{j^1_{x_0} s}(J^1 P) \xrightarrow{(p^1_0)_*} T_{s(x_0)}(P) \to 0.$$

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In that case, R^1 is a generalized Cauchy-Kowaleska system.

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Every hypersurface of M is characteristic for the PDEs system of the extremals of $\mathcal{L}^1_{23}$.
Hence the Euler-Lagrange equations for $\mathcal{L}^1_{23}$ cannot be written in the Cauchy-Kowaleska form, but they are formally integrable and consequently the Cauchy problem admits a solution.

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$p\colon P \to M$ fibred manifold. A PDEs system $R^k \subset J^k(P)$ is formally integrable if

$$p^{k+l}_{k+l-1}\colon R^{k+l} \longrightarrow R^{k+l-1}$$

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then, the PDEs system R^k *is formally integrable.*

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$$\sigma_1 \colon J^1(FM) \times_M \left(S^2 T^* \otimes (\oplus^m T)\right) \longrightarrow T^* \otimes \mathbb{R}^{3(m-2)} \cong \oplus^{3(m-2)} T^*,$$

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(\sigma_1)^{i-3}(j^1_x s, [f]^2_x \otimes v) &= (\omega^1(Y_i) d(X_3 f) - \omega^1(Y_3) d(X_i f))(x), \\
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for $4 \leq i \leq m$, $[f]^2_x \in S^2 T^*$ and $v = ((Y_1)_x, \dots, (Y_m)_x)$.

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As $p^2_1(R^2) = \{j^1_x s \in R^1 \mid \Omega(j^2_1 s) = 0\}$, then $p^2_1\colon R^2 \to R^1$ is surjective if and only if the curvature vanishes.

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The basis $(t_1, \dots, t_m)$ of T_x is quasi-regular for $\mathfrak{g}_1$ at $u = s(x)$ if

$$\dim \mathfrak{g}_{2,u} = \dim \mathfrak{g}_{1,u} + \sum_{j=1}^{m-1} \dim(\mathfrak{g}_{1,u} \cap (T^*_{x,(t_1,\dots,t_j)} \otimes (\oplus^m T_x))).$$

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Let $\sigma\colon FM \times_M (T^* \otimes (\oplus^m T_x)) \longrightarrow M \times \mathbb{R}^{3(m-2)}$ be the symbol of $\varphi\colon J^1(FM) \to M \times \mathbb{R}^{3(m-2)}$.
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Theorem

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Hence, if M and s are of class C^ω, then given a point $j^1_{x_0}\xi_0 \in R^1_s$ there exists a Jacobi field X_ξ defined along s on a neighbourhood of x_0 such that $j^1_{x_0}\xi = j^1_{x_0}\xi_0$.

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